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Four-point functions of different-weight operators in the AdS/CFT correspondence

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Abstract: We calculate four-point correlation functions of two weight-2 and two weight-3 $\frac{1}{2}$-BPS operators in $\mathcal{N}=4$ SYM in the large N limit in supergravity approximation. By the AdS/CFT conjecture, these operators are dual to AdS_5 supergravity scalar fields s_2 and s_3 with mass $m^2=-4$ and $m^2=-3$ respectively. This is the first non-trivial four-point function of mixed-weight operators of lowest conformal dimensions.
We show that the supergravity-induced four-point function splits into a "free" and a "quantum" part, where the quantum contribution obeys non-trivial constraints coming from the insertion procedure in the gauge theory, in particular, it depends on only *one* function of the conformal cross-ratios.

Keywords: AdS-CFT Correspondence, Gauge-gravity correspondence.

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Contents

1. Introduction

Based on investigations of near horizon geometries and scattering from black hole metrics, it was conjectured [1−3] that the large N limit of some superconformal gauge theories in d-dimensional flat space-time is governed by string (supergravity) theories on $d+1$-dimensional anti-de Sitter space (AdS_{d+1}) times a $10-(d+1)$ compact manifold ($\mathcal{M}^{10-(d+1)}$).[1] In particular, strongly coupled $\mathcal{N}=4$ supersymmetric Yang-Mills theory in four dimensions (SYM) is conjectured to be dual to Type IIB supergravity on an $AdS_5 \times S^5$ background.

The compactification of Type IIB supergravity on $AdS_5 \times S^5$ gives rise to an infinite tower of massive Kaluza-Klein (KK) modes in the resulting five-dimensional theory. The isometry group of Type IIB supergravity is identical to the superconformal group of the dual field theory. The kinematical relation between the two theories implies that the scalar KK modes s_k ($k \geq 2$) that are mixtures of the five-form potential and the graviton on S^5, are dual to $\frac{1}{2}$-BPS operators of $\mathcal{N}=4$ SYM. These operators form short superconformal

[1] For a review of the correspondence see e.g. [4].

multiplets and have conformal dimensions that are protected from quantum corrections. By definition, the highest-weight operators of these multiplets are annihilated by half of the Poincaré supercharges.

Another kinematical consequence of the identification mentioned above is that two- and three-point functions of $\frac{1}{2}$-BPS operators are protected from quantum corrections. Therefore, the calculation of the three-point supergravity-induced correlators doesn't give any new dynamical information and one needs to go further to test the conjectured correspondence.

The four-point functions of such operators are not protected from quantum corrections and therefore are the simplest non-trivial candidates to explore the dynamics in the strong coupling limit. Furthermore, the quantum behavior of four-point functions is severely restricted, not only kinematically due to conformal invariance, but also dynamically due to the existence of the Lagrangian of $\mathcal{N} = 4$ SYM. The latter is given by the insertion procedure [5], and reduces the functional freedom predicted by conformal invariance. This inherent dynamical feature of $\mathcal{N} = 4$ SYM has no analogue in Type IIB supergravity and offers the possibility of testing the AdS/CFT correspondence by comparing the supergravity-induced results with the structure predicted by it [6].

The five-dimensional effective action for Type IIB supergravity relevant for the calculation of supergravity-induced four-point functions of $\frac{1}{2}$-BPS operators has already been constructed [7]. The most important feature of the Lagrangian is that quartic couplings have at most four derivatives in the fields. Due to the involved computational work and complexity of the couplings, only specific examples of four-point functions involving four identical operators have been calculated, namely, for the operators with weight $k = 2, 3, 4$ [8–10].[2] It has been found that the supergravity results indeed have the dynamical structure predicted by $\mathcal{N} = 4$ SYM.

In this paper we go beyond and explore the gauge/supergravity duality calculating the first non-trivial four-point function of *mixed* $\frac{1}{2}$-BPS operators with lowest conformal dimensions, i.e., the correlation function involving two $k = 2$ and two $k = 3$ operators. We first establish the general structure predicted by conformal invariance and the insertion procedure, and then compare it with the one calculated from supergravity.

To this end, we first extract the relevant fourth order Lagrangian from the general one. We find that the four-derivatives quartic couplings can be reduced to two and non-derivatives couplings. Hence, the relevant Lagrangian is of *σ-model type*. This characteristic was also found for the relevant Lagrangians necessary for the computation of the four-point functions mentioned above. The cancellation of four derivatives in the present case was indeed expected since the relevant couplings are sub-subextremal [14].

We again find that the amplitude splits into a "free" and a "quantum" part which exactly coincide with the result calculated from $\mathcal{N} = 4$ SYM and the prediction given by the insertion procedure. Since in the supergravity side there is not a quantity analogous to the coupling constant g_{YM}, the latter can be interpreted as another non-trivial check supporting the AdS/CFT correspondence.

[2]For four-point correlators of non-superconformal primary operators, see [11–13].

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The paper is organized as follows: in section 2 the general form of the four-point function under consideration is found from conformal, R- and crossing symmetries. Further constraints on the coefficients are found by the insertion procedure. In section 3, we obtain the Lagrangian that we use to compute the supergravity-induced amplitude. Finally, in section 4 the result obtained from the supergravity analysis is compared with the conformal field theory predictions. Technical details are gathered in the appendices, in particular, the C-algebra (where we include the normalized projectors) and the novel computation of vector and massive symmetric tensor exchange diagrams, where the vector and tensor couple to currents which are not conserved on-shell.

2. Generalities

The conformal structure of SYM restricts the form of the four-point function. In this section the general form of the four-point function of two weight-2 and two weight-3 $\frac{1}{2}$-BPS operators is discussed.

The $\frac{1}{2}$-BPS operators of conformal weight k we consider are single-trace operators in the $\mathcal{N}=4$ SYM theory, given by[3]

$$\mathcal{O}_k^I = C^I_{i_1\cdots i_k}\,\mathrm{tr}(\phi^{i_1}\cdots\phi^{i_k}). \tag{2.1}$$

The fields $\phi^i, i=1,\ldots 6$ are $\mathcal{N}=4$ SYM scalar fields and the $C^I_{i_1,\ldots i_k}$ are traceless symmetric SO(6) tensors (see appendix A). The index I runs over the basis of the corresponding SO(6) irrep with Dynkin labels $[0,k,0]$. We want to find the general structure of the four-point function

$$\langle \mathcal{O}_2^{I_1}\mathcal{O}_2^{I_2}\mathcal{O}_3^{I_3}\mathcal{O}_3^{I_4}\rangle \equiv \langle \mathcal{O}_2^1\mathcal{O}_2^2\mathcal{O}_3^3\mathcal{O}_3^4\rangle$$

and compare it with the result we obtain in the supergravity approximation.

We will apply the methods of sections 2 and 3 of [9] (see also references there) to our specific case. This gives a basis in terms of the tensor structures appearing. There are six different structures belonging to four equivalence classes, which are shown in figure 1. They are called propagator structures, since they appear naturally by connecting propagators of the scalar fields ϕ^i. The elements in an equivalence class are related by crossing symmetry. To get the most general conformally invariant form of the correlator, we multiply each structure by a function of the conformal cross-ratios

$$s=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2},\qquad t=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}, \tag{2.2}$$

[3]For $k\geq 4$ the scalar fields s_k correspond to *extended* $\frac{1}{2}$-BPS operators, where these single-trace operators receive a multi-trace correction. For regular correlators however, this operator mixing is suppressed in the large N limit [15, 14].

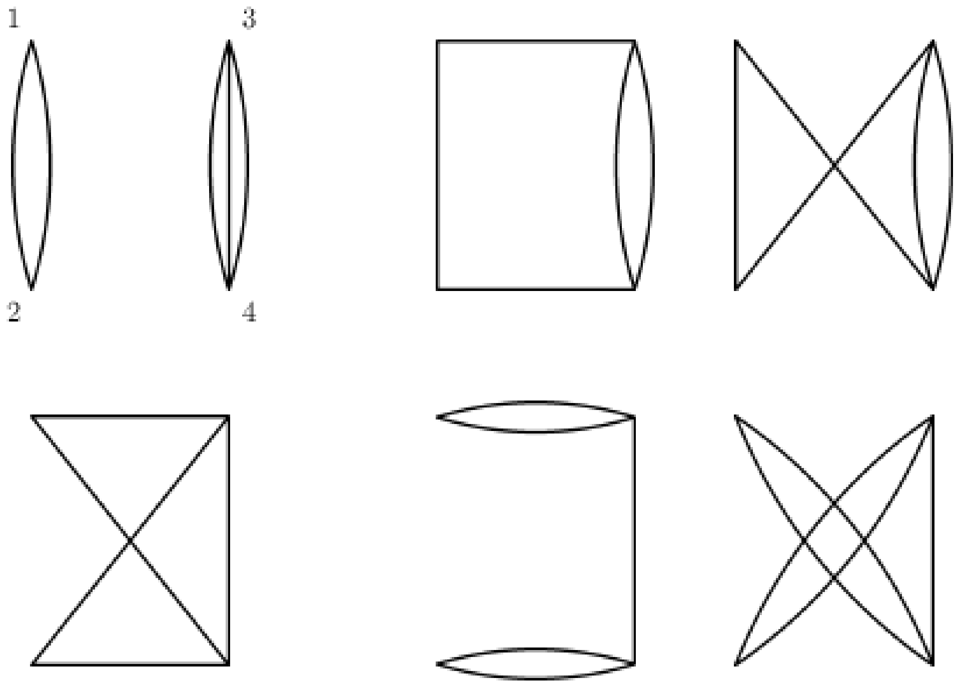

Figure 1: Propagator structures. The diagrams are divided into four equivalence classes. The elements in an equivalence class are related by crossing symmetry.

where $x_{ij}^2 = |\vec{x}_i - \vec{x}_j|^2$. Following this procedure gives, in terms of the propagator basis from appendix A,

$$\begin{aligned}\langle O_2^1(x_1)O_2^2(x_2)O_3^3(x_3)O_3^4(x_4)\rangle = a(s,t)\frac{\delta_2^{12}\delta_3^{34}}{x_{12}^4 x_{34}^6} + b(s,t)\frac{S^{1234}}{x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2} \\ +c_1(s,t)\frac{C^{1243}}{x_{12}^2 x_{34}^4 x_{13}^2 x_{24}^2} + c_2(s,t)\frac{C^{1234}}{x_{12}^2 x_{34}^4 x_{14}^2 x_{23}^2} \\ +d_1(s,t)\frac{\Upsilon^{1234}}{x_{13}^4 x_{34}^2 x_{24}^4} + d_2(s,t)\frac{\Upsilon^{1243}}{x_{34}^2 x_{23}^4 x_{14}^4}.\end{aligned} \tag{2.3}$$

Here, the O_k denote the canonically normalized versions of the operators $\mathcal{O}_k$, such that $\langle O_k^{I_1}(x_1)O_k^{I_2}(x_2)\rangle = \frac{\delta^{I_1 I_2}}{x_{12}^{2k}}$. This is the most general form, allowed by conformal- and R-symmetry, of the four-point function we consider. The coefficient functions in eq. (2.3) split into a "quantum" and a "free" part. We will denote the quantum part with a hat, e.g. $a(s,t) = \hat{a}(s,t) + a$, where a is the free part discussed at the end of this section.

By simple symmetry considerations, relations between the coefficient functions can be derived. Note that when permuting $x_1 \leftrightarrow x_2$, the cross-ratios transform as $s \to s/t$, and $t \to 1/t$. On the other hand, in the r.h.s. of eq. (2.3), this corresponds to interchanging the representation labels 1 and 2. Using the symmetry properties of the C-tensors, it then immediately follows that

$$\begin{aligned} a(s,t) &= a(s/t, 1/t), & b(s,t) &= b(s/t,1/t), \\ c_1(s/t,1/t) &= c_2(s,t), & d_1(s/t,1/t) &= d_2(s,t). \end{aligned} \tag{2.4}$$

These are the crossing symmetry relations.

There is another relation between these coefficient functions. This is based on the insertion formula [5]. This procedure gives additional constraints on the *quantum* part of the correlator. From the results of section 3 of ref. [9], we find that the coefficient functions can be expressed in terms of a *single* function $\mathcal{F}(s,t)$ by

$$\hat{a}(s,t) = s\mathcal{F}(s,t), \quad \hat{d}_1(s,t) = \mathcal{F}(s,t), \quad \hat{d}_2(s,t) = t\mathcal{F}(s,t), \tag{2.5}$$

and

$$\begin{aligned} \hat{b}(s,t) &= (s-t-1)\mathcal{F}(s,t), \\ \hat{c}_1(s,t) &= (t-s-1)\mathcal{F}(s,t), \\ \hat{c}_2(s,t) &= (1-s-t)\mathcal{F}(s,t). \end{aligned} \tag{2.6}$$

The crossing symmetry relations (2.4) then imply that $\mathcal{F}(s/t, 1/t) = t\mathcal{F}(s,t)$. Hence, all dynamical information in the four-point function is completely determined by the single function $\mathcal{F}(s,t)$ of the cross-ratios. This result is purely based on conformal field theory considerations and the insertion procedure. Later we will compare the result from supergravity calculations with this general form.

Finally, the coefficient functions in free field theory are, in the large N limit, given by

$$a = 1, \quad b = \frac{12}{N^2}, \quad c_{1,2} = \frac{6}{N^2}, \quad d_{1,2} = 0. \tag{2.7}$$

Note that the color structure of the diagrams implies that the free field contribution from *one-particle reducible* diagrams identically vanishes. Also, the coefficient a comes from a disconnected diagram, and hence doesn't appear when considering the connected four-point function.

3. Supergravity Lagrangian

The computation of four-point functions of $\frac{1}{2}$-BPS operators in supergravity approximation requires the $5d$ effective quartic action of compactified Type IIB supergravity on an $AdS_5 \times S^5$ background, and the identification of the relevant parts. The effective five-dimensional action can be written as [7]

$$S = \frac{N^2}{8\pi^2} \int \mathrm{d}^5 z \sqrt{g} (\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4), \tag{3.1}$$

a sum of quadratic, cubic and quartic terms. In comparison with ref. [7], we will work on the Euclidean version of AdS_5, which results in an overall minus sign, and rescale the fields to match the quadratic part presented below. In equation (3.1), g is the determinant of the Euclidean metric on AdS_5, $\mathrm{d}s^2 = \frac{1}{z_0^2}(\mathrm{d}z_0^2 + \mathrm{d}x^i \mathrm{d}x^i)$, with $i = 1, \ldots, 4$.

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The quadratic terms were found in [16]. The relevant part in this case is

$$
\begin{aligned}
\mathcal{L}_2 =& \frac{1}{4}\left(\nabla_\mu s_2^1 \nabla^\mu s_2^1 - 4 s_2^1 s_2^1\right) + \frac{1}{4}\left(\nabla_\mu s_3^1 \nabla^\mu s_3^1 - 3 s_3^1 s_3^1\right) \\
&+ \frac{1}{2}(F^1_{\mu\nu,1})^2 + \frac{1}{2}(F^1_{\mu\nu,2})^2 + 3(A^1_{\mu,2})^2 \\
&+ \frac{1}{4}\nabla_\rho \phi_{\mu\nu,0} \nabla^\rho \phi_0^{\mu\nu} - \frac{1}{2}\nabla_\mu \phi_{\mu\rho,0} \nabla^\nu \phi_{\nu\rho,0} + \frac{1}{2}\nabla_\mu \phi^\rho_{\rho,0} \nabla_\nu \phi_0^{\mu\nu} - \frac{1}{4}\nabla_\rho \phi^\mu_{\mu,0} \nabla^\rho \phi^\nu_{\nu,0} \\
&- \frac{1}{2}\phi_{\mu\nu,0}\phi_0^{\mu\nu} + \frac{1}{2}(\phi^\nu_{\nu,0})^2 \\
&+ \frac{1}{4}\nabla_\rho \phi^1_{\mu\nu,1} \nabla^\rho \phi_1^{\mu\nu\,1} - \frac{1}{2}\nabla_\mu \phi^1_{\mu\rho,1} \nabla^\nu \phi^1_{\nu\rho,1} + \frac{1}{2}\nabla_\mu \phi^{\rho\,1}_{\rho,1} \nabla_\nu \phi_1^{\mu\nu\,1} - \frac{1}{4}\nabla_\rho \phi^{\mu\,1}_{\mu,1} \nabla^\rho \phi^{\nu\,1}_{\nu,1} \\
&+ \frac{3}{4}\phi^1_{\mu\nu,1}\phi_1^{\mu\nu\,1} - \frac{7}{4}(\phi^{\nu\,1}_{\nu,1})^2,
\end{aligned}
\tag{3.2}
$$

where $F_{\mu\nu,k} = \partial_\mu A_{\nu,k} - \partial_\nu A_{\mu,k}$, and a summation over the upper indices, which run over the basis of the irrep corresponding to the field, is implied. A novelty in these calculations is the appearance of the vector $A_{\mu,1}$ and the tensor $\phi_{\mu\nu,1}$. These appear coupled to one s_2 and one s_3 scalar field, which lead to tree diagrams where these fields are exchanged. In previous calculations, these fields were not present because of SO(6) selection rules.

The cubic couplings needed to calculate four-point functions of arbitrary $\frac{1}{2}$-BPS operators were calculated in refs. [17, 15, 18]. The couplings are given in terms of C-tensors, which are related to spherical harmonics on S^5. They are the Clebsch-Gordan coefficients for tensor products of two SO(6) irreps. For definitions and normalization, we refer to appendix B of ref. [9]. We will employ the notation

$$
\langle C^1_{k_1} C^2_{k_2} C^3_{[a_1,a_2,a_3]} \rangle
$$

where lower indices k_i denote the irrep with Dynkin labels $[0, k_i, 0]$, and the upper indices run over the basis of the corresponding irrep. With this notation the cubic couplings become

$$
\begin{aligned}
\mathcal{L}_3 =& -\frac{1}{3}\langle C^1_2 C^2_2 C^3_{[0,2,0]}\rangle s_2^1 s_2^2 s_2^3 - 3\langle C^1_3 C^2_3 C^3_{[0,2,0]}\rangle s_3^1 s_3^2 s_2^3 \\
&- \frac{1}{4}\left(\nabla^\mu s_2^1 \nabla^\nu s_2^1 \phi_{\mu\nu,0} - \frac{1}{2}\left(\nabla^\mu s_2^1 \nabla_\nu s_2^1 - 4 s_2^1 s_2^1\right)\phi^\nu_{\nu,0}\right) \\
&- \frac{1}{4}\left(\nabla^\mu s_3^1 \nabla^\nu s_3^1 \phi_{\mu\nu,0} - \frac{1}{2}\left(\nabla^\mu s_3^1 \nabla_\nu s_3^1 - 3 s_3^1 s_3^1\right)\phi^\nu_{\nu,0}\right) \\
&- \frac{1}{2}\langle C^1_2 C^2_3 C^3_{[0,1,0]}\rangle \left(\nabla^\mu s_2^1 \nabla^\nu s_3^2 \phi^3_{\mu\nu,1} + \frac{1}{2}\left(\nabla^\mu s_2^1 \nabla_\mu s_3^2 - 6 s_2^1 s_3^2\right)\phi^{\nu 3}_{\nu,1}\right) \\
&- \langle C^1_2 C^2_2 C_{[1,0,1]}\rangle s_2^1 \nabla^\mu s_2^2 A^3_{\mu,1} - \frac{3}{2}\langle C^1_3 C^2_3 C^3_{[1,0,1]}\rangle s_3^1 \nabla^\mu s_3^2 A^3_{\mu,1} \\
&- \sqrt{3}\langle C^1_2 C^2_3 C^3_{[1,1,1]}\rangle s_2^1 \nabla^\mu s_3^2 A^3_{\mu,2} - \sqrt{3}\langle C^1_3 C^2_2 C^3_{[1,1,1]}\rangle s_3^1 \nabla^\mu s_2^2 A^3_{\mu,2}.
\end{aligned}
\tag{3.3}
$$

One interesting feature one can read off from the Lagrangian is the appearance of exchange diagrams of a vector and a symmetric tensor field which do not couple to conserved currents: the weight of the external scalar fields are different.

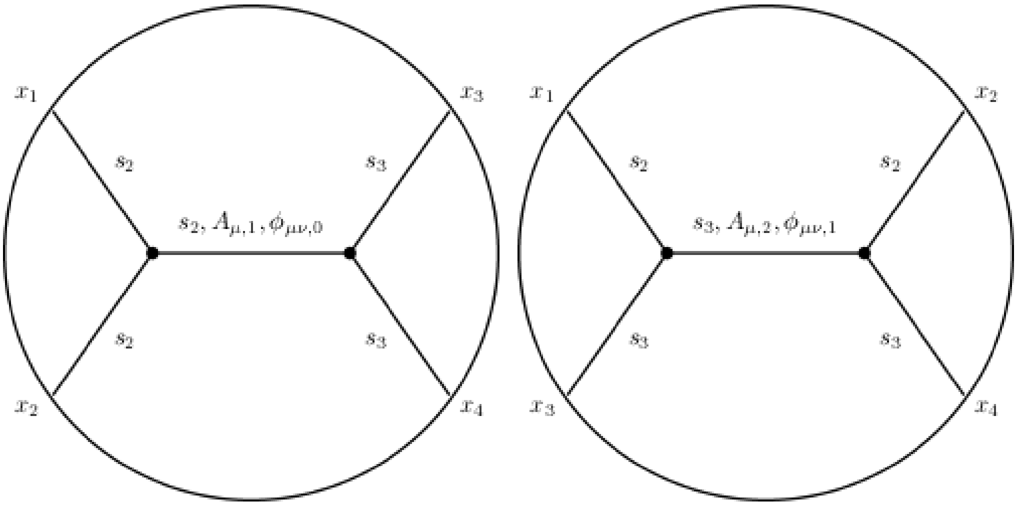

Figure 2: Witten diagrams contributing to the four-point function.

The quartic couplings are the hardest to compute. The result (the details are in appendix B) is

$$\mathcal{L}_4 = -\frac{1}{4}\left(C^{1234} - S^{1234}\right) s_2^1 \nabla_\mu s_2^2 s_3^3 \nabla^\mu s_3^4 + \frac{3}{8}\left(9C^{1234} + 5S^{1234} - \delta_2^{12}\delta_3^{34} - 3\Upsilon^{1234}\right) s_2^1 s_2^2 s_3^3 s_3^4.$$

Now that the Lagrangian has been found, we can determine its on-shell value. This amounts to calculating exchange (and contact) diagrams. Witten diagrams of all exchange integrals contributing are shown in figure 2. In comparison with previous results, we also have to compute the exchange of a massive vector and a massive tensor coupled to currents which aren't conserved on-shell. The method we use to calculate them is described in appendix D.

4. Verifying CFT predictions and conclusions

After the on-shell value of the Lagrangian is calculated, we can trivially find the four-point function. After finding the coefficient functions in terms of D-functions, we express them in terms of the cross-ratios s and t by means of the $\overline{D}$-functions (see appendix C). We find that the four-point function has indeed the structure of eq. (2.3).

There are two ways to verify the relations predicted by the insertion procedure. First of all, one could work with the $\overline{D}$-functions, and use the identities in appendix D.2 of [9]. Crossing symmetry and other relations between $\overline{D}$-functions are listed there. An advantage of this method is that it is possible to obtain a simple form of the coefficient functions. In the present situation, however, the above method becomes rather cumbersome and error-prone, due to the large number of different $\overline{D}$-functions involved. A more straightforward method is to express $\overline{D}$-functions in terms of differential operators $\overline{\mathbf{D}}$, defined in appendix C. Using equations (C.6), it is then possible to write the coefficients completely in terms of $\ln s, \ln t$ and $\Phi(s,t)$. Using that $\Phi(s/t, 1/t) = t\Phi(s,t)$, one finds that the crossing symmetries (2.4) are indeed satisfied. This however is a rather trivial check, as these crossing symmetries follow automatically if one considers the correct permutations when calculating the individual contributions to the four-point function.

To check the insertion procedure predictions, we compare the connected four-point function obtained from the supergravity result with the connected four-point function from the CFT. From the coefficient $a(s,t)$, we find the single function $\mathcal{F}(s,t)$:

$$\mathcal{F}(s,t) = \frac{3}{N^2}\left(-\overline{D}_{1133}(s,t) + 4s\overline{D}_{2233}(s,t) + (1-s+t)\overline{D}_{2244}(s,t)\right),$$

where $a(s,t) = \hat{a}(s,t) = s\mathcal{F}(s,t)$.

Then, by using the second method described above, we find

$$\begin{aligned} b(s,t) - (s-t-1)\mathcal{F}(s,t) &= \frac{12}{N^2} \\ c_1(s,t) - (t-s-1)\mathcal{F}(s,t) &= \frac{6}{N^2} \\ d_1(s,t) - \mathcal{F}(s,t) &= 0, \end{aligned} \tag{4.1}$$

and similar for c_2 and d_2. But the r.h.s. of (4.1) are nothing but the free field contributions from eq. (2.7)! Moreover, by comparing with eqs. (2.5) and (2.6), we see that the coefficients indeed satisfy the restrictions obtained by the insertion procedure, up to those constants. Hence, we conclude that the coefficients calculated from supergravity split into a quantum and a free field part, where the latter is due to *one-particle irreducible* (1PI) diagrams.

The novelty of this result is that it is the first time four-point correlators of $\frac{1}{2}$-BPS operators of different weight (apart from some trivial (extremal) cases) have been calculated from Type IIB supergravity. However, we do find the same features as for the case where all operators are of equal weight $k = 2, 3, 4$; namely, that the quantum part of the supergravity-induced amplitude satisfies the constrains obtained from the insertion procedure in the field theory side, reducing the functional freedom to one (or two in the case $k = 4$) function of the conformal cross-ratios. This gives further support for the AdS/CFT correspondence, since there is no obvious explanation on the supergravity side for these results.

It would be interesting to compare our findings with the ones obtained from perturbative SYM, since both results correspond to a different regime. Furthermore, one can find the OPE of operators involved in this correlator, to compute anomalous dimensions [19–22].

Another interesting question is to understand better the implications of the insertion procedure in CFT on the supergravity side. These considerations could possibly lead to a simpler description of the AdS_5 supergravity. One could also include corrections to the supergravity result, for example by considering D-instanton effects.

Acknowledgments

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A. C-algebra

The couplings of the five-dimensional action are described in terms of integrals of spherical harmonics on S^5. These spherical harmonics can be described by C-tensors $C^I_{i_1 \dots i_k}$. The C-tensor transforms in the $[0, k, 0]$ irrep of SO(6) (see the appendices of [17, 15]). These C-tensors are symmetrized according to the corresponding Young pattern of the irrep they correspond to. The integrals of the spherical harmonics appearing in the couplings are basically the Clebsch-Gordan coefficients of products of two SO(6) irreps, and can be expressed in terms of SO(6) rank 3 tensors

$$\langle C^1_{k_1} C^2_{k_2} C^3_{[a_1,a_2,a_3]} \rangle,$$

where we denote the representation index I_1 by 1, etc. These rank 3 tensors are constructed by contracting particular subset of indices of the C-tensors $C^{I_1}_{[0,k_1,0]}$, $C^{I_2}_{[0,k_2,0]}$ and $C^{I_3}_{[a_1,a_2,a_3]}$. See appendix B of ref. [9] for details.

The quartic couplings are described in terms of products of two Clebsch-Gordan coefficients, that arise from overlapping intergrals of spherical harmonics

$$\langle C^1_{k_1} C^2_{k_2} C^5_{[a_1,a_2,a_3]} \rangle \langle C^3_{k_3} C^4_{k_4} C^5_{[a_1,a_2,a_3]} \rangle. \tag{A.1}$$

Since there is a summation over the representation index I_5, we can use a completeness relation to express this tensor in the so-called propagator basis. In these calculations, the expressions in the appendix of ref. [7] are helpful. One can refer to [9] for an example of their application.

The procecudure outlined above yields four different kinds of rank 4 tensors. One should notice that for each kind, there are two C-tensors transforming as $[0, 2, 0]$, and two as $[0, 3, 0]$, involved. This should be taken into account carefully when permuting the representation indices. The tensor structures involved are given by

$$\begin{aligned}
\delta_2^{12} \delta_3^{34} &= C^1_{ij} C^2_{ij} C^3_{klm} C^4_{klm}, && \text{(A.2)}\\
C^{1234} &= C^1_{ij} C^2_{jk} C^3_{klm} C^4_{ilm}, && \text{(A.3)}\\
\Upsilon^{1234} &= C^1_{ij} C^2_{lm} C^3_{ijk} C^4_{lmk}, && \text{(A.4)}\\
S^{1234} &= C^1_{ik} C^2_{jl} C^3_{lkm} C^4_{ijm}. && \text{(A.5)}
\end{aligned}$$

The tensors $\delta_2^{12} \delta_3^{34}$ and S^{1234} are symmetric under $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ separately, while C^{1234} and Υ^{1234} have the following symmetry relations:

$$C^{1234} = C^{2143}, \quad \Upsilon^{1234} = \Upsilon^{2143}. \tag{A.6}$$

We now evaluate eq. (A.1) for the cases of interest for us. In the case of $k_1 = k_2 = 2, k_3 = k_4 = 3$, this gives the following results. Note that the selection rules for k_5 allow only for these values of k_5.

$$
\begin{aligned}
\langle C_2^1C_2^2C_{[0,0,0]}^5\rangle\langle C_3^3C_3^4C_{[0,0,0]}^5\rangle &= \delta_2^{12}\delta_3^{34},\\
\langle C_2^1C_2^2C_{[0,2,0]}^5\rangle\langle C_3^3C_3^4C_{[0,2,0]}^5\rangle &= \frac{1}{2}C^{1234}+\frac{1}{2}C^{1243}-\frac{1}{6}\delta_2^{12}\delta_3^{34},\\
\langle C_2^1C_2^2C_{[0,4,0]}^5\rangle\langle C_3^3C_3^4C_{[0,4,0]}^5\rangle &= -\frac{2}{15}C^{1234}-\frac{2}{15}C^{1243}+\frac{2}{3}S^{1234}\\
&\quad +\frac{1}{6}\Upsilon^{1243}+\frac{1}{6}\Upsilon^{1234}+\frac{1}{60}\delta_2^{12}\delta_3^{34}.
\end{aligned}
\tag{A.7}
$$

It is interesting to see that they are of the same form as those found in the cases where C^1, C^2, C^3 and C^4 are all in the same representation [10, 9], if one makes the proper identifications.

For the summation over the vector representation we get

$$
\begin{aligned}
\langle C_2^1C_2^2C_{[1,0,1]}^5\rangle\langle C_3^3C_3^4C_{[1,0,1]}^5\rangle &= 2(C^{1243}-C^{1234}),\\
\langle C_2^1C_2^2C_{[1,2,1]}^5\rangle\langle C_3^3C_3^4C_{[1,2,1]}^5\rangle &= \frac{1}{3}(C^{1234}-C^{1243})+\frac{2}{3}(\Upsilon^{1234}-\Upsilon^{1243}).
\end{aligned}
\tag{A.8}
$$

And the tensor representation

$$
\begin{aligned}
\langle C_2^1C_2^2C_{[2,0,2]}^5\rangle\langle C_3^3C_3^4C_{[2,0,2]}^5\rangle = &-\frac{2}{3}\left(C^{1234}+C^{1243}\right)+\frac{4}{3}\left(\Upsilon^{1234}+\Upsilon^{1243}\right)\\
&-\frac{8}{3}S^{1234}+\frac{2}{15}\delta_2^{12}\delta_3^{34}.
\end{aligned}
\tag{A.9}
$$

Next we consider the case where $k_1 = k_3 = 2$ and $k_2 = k_4 = 3$. These are distinctively different from the cases encountered in previous work, since the selection rules now give other values for the representation that is summed over. For summation over the scalar representation the results are

$$
\begin{aligned}
\langle C_2^1C_3^2C_{[0,1,0]}^5\rangle\langle C_2^3C_3^4C_{[0,1,0]}^5\rangle &= \Upsilon^{1324},\\
\langle C_2^1C_3^2C_{[0,3,0]}^5\rangle\langle C_2^3C_3^4C_{[0,3,0]}^5\rangle &= \frac{1}{3}C^{1342}+\frac{2}{3}S^{1324}-\frac{1}{6}\Upsilon^{1324},\\
\langle C_2^1C_3^2C_{[0,5,0]}^5\rangle\langle C_2^3C_3^4C_{[0,5,0]}^5\rangle &= \frac{3}{5}C^{1324}-\frac{1}{10}C^{1342}-\frac{1}{5}S^{1324}\\
&\quad +\frac{1}{10}\delta_2^{13}\delta_3^{24}+\frac{3}{10}\Upsilon^{1342}+\frac{1}{50}\Upsilon^{1324},
\end{aligned}
\tag{A.10}
$$

and for the vector cases

$$
\begin{aligned}
\langle C_2^1C_3^2C_{[1,1,1]}^5\rangle\langle C_2^3C_3^4C_{[1,1,1]}^5\rangle &= \frac{3}{2}C^{1342}-\frac{3}{2}S^{1324}-\frac{3}{10}\Upsilon^{1324},\\
\langle C_2^1C_3^2C_{[1,3,1]}^5\rangle\langle C_2^3C_3^4C_{[1,3,1]}^5\rangle &= \frac{5}{12}C^{1324}-\frac{25}{84}C^{1324}+\frac{5}{21}S^{1324}\\
&\quad +\frac{5}{12}\delta_2^{13}\delta_3^{24}+\frac{1}{42}\Upsilon^{1324}-\frac{5}{6}\Upsilon^{1342}.
\end{aligned}
\tag{A.11}
$$

Finally, the tensor case gives

$$
\begin{aligned}
\langle C_2^1C_3^2C_{[2,1,2]}^5\rangle\langle C_2^3C_3^4C_{[2,1,2]}^5\rangle &= -\frac{16}{9}C^{1324}-\frac{40}{63}C^{1342}-\frac{16}{63}S^{1324}\\
&\quad +\frac{8}{9}\delta_2^{13}\delta_3^{24}+\frac{8}{63}\Upsilon^{1324}+\frac{8}{9}\Upsilon^{1342}.
\end{aligned}
\tag{A.12}
$$

Tensor	$\delta_2^{12}\delta_3^{34}$	C^{1234}	C^{1243}	Υ^{1234}	Υ^{1243}	S^{1234}
$\delta_2^{12}\delta_3^{34}$	1000	$\frac{500}{3}$	$\frac{500}{3}$	50	50	$\frac{50}{3}$
C^{1234}	$\frac{500}{3}$	$\frac{725}{3}$	$\frac{25}{3}$	$\frac{25}{3}$	125	$\frac{25}{2}$
Υ^{1234}	50	$\frac{25}{3}$	125	$\frac{1250}{3}$	$\frac{25}{3}$	$\frac{125}{3}$
S^{1234}	$\frac{50}{3}$	$\frac{25}{2}$	$\frac{25}{2}$	$\frac{125}{3}$	$\frac{125}{3}$	$\frac{425}{4}$

Table 1: Pairings between independent tensors.

The results of the remaining cases are the same, if one changes the representation labels accordingly, except for equation (A.11), which acquires an additional minus sign in the cases $k_1 = k_4 = 3, k_2 = k_3 = 2$ and $k_1 = k_4 = 2, k_2 = k_3 = 3$.

Finally, for completeness and in order to facilitate a future analysis of the OPE of the four-point function, we include a table of pairings between the elements of the propagator basis. These can be found using the completeness relation. Our results are summarized in table 1.

With these pairings we can fix the normalization for the projectors $P^{1234}_{[k_1,k_2,k_3]}$ to be $P^{1234}_{[k_1,k_2,k_3]}P^{1234}_{[k_1,k_2,k_3]} = \dim[k_1, k_2, k_3]$, and check that they are orthogonal. These projectors can be used to project the four-point correlator onto the irreps appearing in the tensor decompositions of $[0, 2, 0] \times [0, 3, 0]$ and $[0, 2, 0] \times [0, 2, 0]$. The projectors itself we have already found: they are proportional to the summations of C-tensors we found above. For example, $P^{1234}_{[0,1,0]} \propto \langle C^1_2 C^3_3 C^5_{[0,1,0]}\rangle\langle C^2_2 C^4_3 C^5_{[0,1,0]}\rangle$. With table 1 it is trivial to normalize them properly and check orthogonality.

B. Reduction of quartic couplings

In this section we describe how we can rewrite the four-derivative terms in the Lagrangian, and end up with a remarkably simple expression for the quartic contribution to the Lagrangian. We show that the Lagrangian is of *σ-model type*, a feature that was also found for the effective Lagrangian in calculations of equal-weight $k = 2, 3$ and 4 four-point functions [8 − 10]. The fact that the Lagrangian reduces to a simple expression in each case, suggests that there may be a simpler description.

The quartic couplings of the scalar fields are given in [7]. They can be written as

$$\begin{aligned} \mathcal{L}^4 = \mathcal{L}^{(4)I_1I_2I_3I_4}_{k_1k_2k_3k_4} s^{I_1}_{k_1}\nabla_\mu s^{I_2}_{k_2}\nabla^\nu\nabla_\nu(s^{I_3}_{k_3}\nabla^\mu s^{I_4}_{k_4}) \\ +\mathcal{L}^{(2)I_1I_2I_3I_4}_{k_1k_2k_3k_4} s^{I_1}_{k_1}\nabla_\mu s^{I_2}_{k_2} s^{I_3}_{k_3}\nabla^\mu s^{I_4}_{k_4} + \mathcal{L}^{(0)I_1I_2I_3I_4}_{k_1k_2k_3k_4} s^{I_1}_{k_1} s^{I_2}_{k_2} s^{I_3}_{k_3} s^{I_4}_{k_4}, \end{aligned} \tag{B.1}$$

where in the case we are interested in, two of the k_i's are equal to 2, and the other two are equal to 3. Hence, this allows for 6 possible permutations. The indices I_i run over the basis of the representation $[0, k_i, 0]$, and should be summed over. From now on, we will denote this indices simply as superscript 1,2,3 and 4.

We now proceed as in refs. [9, 10]. We re-expand the products Clebsch-Gordan coefficients in the couplings, which form the so-called "OPE basis", given by

$$\langle C^1_{k_1} C^2_{k_2} C^5_{[a_1,a_2,a_3]}\rangle\langle C^3_{k_3} C^4_{k_4} C^5_{[a_1,a_2,a_3]}\rangle,$$

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over the propagator basis. This is done in appendix A. The subscripts k_i denote that the C-tensor transforms according to the $[0,k_i,0]$ irrep.

B.1 Four-derivative couplings

In the four-derivative couplings we encounter two basic tensor structures. We will denote them as

$$\begin{aligned} A^{1234} &= \frac{3}{5\cdot 2^{16}}\left(41(C^{1234}-C^{1243})+31(\Upsilon^{1234}-\Upsilon^{1243})\right) \\ \Sigma^{1234} &= \frac{9}{5\cdot 2^{18}}\left(6(C^{1234}+C^{1243})+12S^{1234}+\delta_2^{12}\delta_3^{34}+3(\Upsilon^{1234}+\Upsilon^{1243})\right). \end{aligned} \tag{B.2}$$

Note that A^{1234} is anti-symmetric under $3\leftrightarrow 4$, while Σ^{1234} is symmetric. With this notation, the four-derivative couplings become

$$\begin{aligned} \mathcal{L}_4^{(4)} = &-\left[A^{1234}+\Sigma^{1234}\right] s_2^1\nabla_\mu s_2^2\nabla\cdot\nabla(s_3^3\nabla^\mu s_3^4) \\ &-\left[A^{3412}+\Sigma^{3412}\right] s_3^1\nabla_\mu s_3^2\nabla\cdot\nabla(s_2^3\nabla^\mu s_2^4) \\ &+\left[A^{1324}+\Sigma^{1324}\right] s_2^1\nabla_\mu s_3^2\nabla\cdot\nabla(s_2^3\nabla^\mu s_3^4) \\ &+\left[A^{2413}+\Sigma^{2413}\right] s_3^1\nabla_\mu s_2^2\nabla\cdot\nabla(s_3^3\nabla^\mu s_2^4) \\ &+A^{4123}s_2^1\nabla_\mu s_3^2\nabla\cdot\nabla(s_3^3\nabla^\mu s_2^4) \\ &+A^{3214}s_3^1\nabla_\mu s_2^2\nabla\cdot\nabla(s_2^3\nabla^\mu s_3^4) \end{aligned} \tag{B.3}$$

To rewrite these terms, we first note that on an AdS_5 background, we have the following important formula, obtained by explicitly writing out the $\nabla\cdot\nabla$ derivative in the four-derivative terms, and use that

$$[\nabla_\mu,\nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma,$$

where $R^\rho{}_{\sigma\mu\nu}$ is the Riemann tensor associated with the Euclidean metric on AdS_5. Applying this formula leads to $\nabla^2\nabla_\mu s_k^{I_k} = \nabla_\mu(\nabla^2-4)s_k^{I_k}$. Using this result we find

$$\begin{aligned} &s_{k_1}^1\nabla_\mu s_{k_2}^2\nabla\cdot\nabla(s_{k_3}^3\nabla^\mu s_{k_4}^3) = \\ &\qquad (m_{k_3}^2+m_{k_4}^2-4)s_{k_1}^1\nabla_\mu s_{k_2}^2 s_{k_3}^3\nabla^\mu s_{k_4}^4 + 2s_{k_1}^1\nabla_\mu s_{k_2}^2\nabla_\nu s_{k_3}^3\nabla^\nu\nabla^\mu s_{k_4}^4, \end{aligned} \tag{B.4}$$

where the equations of motion for the s-fields on an AdS-background are used.[4]

If we now use formula (B.4) on each line in eq. (B.3), it is easy to see, after relabeling the summation indices, that the four-derivative terms in the first four lines cancel each other. For the last two lines, recall that A^{1234} is anti-symmetric under the interchange of 3 and 4. But $s_2^1\nabla_\mu s_3^2\nabla_\nu s_3^3\nabla^\mu\nabla^\nu s_3^4$ is symmetric under $2\leftrightarrow 3$, hence $A^{4123}s_2^1\nabla_\mu s_3^2\nabla_\nu s_3^3\nabla^\mu\nabla^\nu s_3^4$ must vanish. A similar argument holds for the last line, hence we conclude that *all four-derivative terms vanish.*

[4]The equation of motion we use is $(\nabla^2-m_k^2)s_k=0$, that is, we do not include the correction terms, since they do not give contributions to the four-point function. This is because these correction terms include the fields on AdS_5, hence this would lead to terms with five fields or more.

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If we sum and relabel the remaining terms after applying equation (B.4), we see that the four-derivative couplings give a contribution to the two-derivative couplings, given by

$$\begin{aligned}\mathcal{L}_4^{(4)} = \Sigma^{1234}(m_2^2+m_3^2-4)(&-2s_2^1\nabla_\mu s_2^2 s_3^3\nabla^\mu s_3^4 \\ &+ s_2^1\nabla_\mu s_3^3 s_2^2\nabla^\mu s_3^4 + s_3^3\nabla_\mu s_2^1 s_3^4\nabla^\mu s_2^2),\end{aligned} \tag{B.5}$$

since the terms A^{1234} times an expression symmetric in $3 \leftrightarrow 4$ vanish.

We can simplify this even further. Notice that by a partial integration[5] we have for a general tensor χ^{1234}

$$\chi^{1234}s_2^1\nabla_\mu s_3^3 s_2^2\nabla^\mu s_3^4 = -(\chi^{2134}+\chi^{1234})s_2^1\nabla_\mu s_2^2 s_3^3\nabla^\mu s_3^4 - m_3^2 s_2^1 s_2^2 s_3^3 s_3^4, \tag{B.6}$$

where the linearized equation of motion is used again. If we use this in eq. (B.5) and the symmetry properties of the tensor Σ^{1234}, it reduces to

$$\mathcal{L}_4^{(4)} = \Sigma^{1234}(m_2^2+m_3^2-4)(-6s_2^1\nabla_\mu s_2^2 s_3^3\nabla^\mu s_3^4 - (m_2^2+m_3^2)s_2^1 s_2^2 s_3^3 s_3^4). \tag{B.7}$$

This completes the calculation of the four-derivative couplings.

B.2 Two-derivative couplings

Proceeding in the same manner as before, we find for the two-derivative couplings

$$\begin{aligned}\mathcal{L}_4^{(2)} = &-\frac{1}{165150720}[337869022C^{1234} + 296581342C^{1243} - 249247719\delta^{12}\delta^{34} \\ &\qquad + 861160876S^{1234} + 62232811(\Upsilon^{1234}+\Upsilon^{1243})]s_2^1\nabla_\mu s_2^2 s_3^3\nabla^\mu s_3^4 \\ &-\frac{1}{660602880}[62281822C^{1234} + 529982734C^{1243} - 249995634\delta^{12}\delta^{34} \\ &\qquad + 901550428S^{1234} - 56591825\Upsilon^{1234} \\ &\qquad + 180608383\Upsilon^{1243}](s_3^3\nabla_\mu s_2^1 s_3^4\nabla^\mu s_2^2 + s_2^1\nabla_\mu s_3^3 s_2^2\nabla^\mu s_3^4)\end{aligned}$$

If we now use eq. (B.6) again, we can rewrite this as

$$\begin{aligned}\mathcal{L}_4^{(2)} = &-\frac{1}{655360}[165622C^{1234} + 1782C^{1243} + 297\delta^{12}\delta^{34} \\ &\quad - 160276S^{1234} + 891(\Upsilon^{1234}+\Upsilon^{1243})]s_2^1\nabla_\mu s_2^2 s_3^3\nabla^\mu s_3^4 \\ &+\frac{(m_3^2+m_2^2)}{660602880}[62281822C^{1234} + 529982734C^{1243} - 249995634\delta^{12}\delta^{34} \\ &\quad + 901550428S^{1234} - 56591825\Upsilon^{1234} + 180608383\Upsilon^{1243}]s_2^1 s_2^2 s_3^3 s_3^4.\end{aligned} \tag{B.8}$$

B.3 Non-derivative couplings

The non-derivative terms contribute, after using the symmetry under $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$, by

$$\begin{aligned}\mathcal{L}_4^{(0)} = \frac{1}{94371840}\big[&911368268C^{1234} + 1079096380S^{1234} - 60339107\delta_2^{12}\delta_3^{34} \\ &+ 18147614\Upsilon^{1234}\big]s_2^1 s_2^2 s_3^3 s_3^4.\end{aligned}$$

[5]The boundary terms do not contribute to the four-point function, see [14].

A remarkable thing happens when we now add all quartic terms: all the "ugly" coefficients add up to a very simple result

$$\begin{aligned}\mathcal{L}_4 = &-\frac{1}{4}\left(C^{1234} - S^{1234}\right) s_2^1 \nabla_\mu s_2^2 s_3^3 \nabla^\mu s_3^4 \\ &+ \frac{3}{8}\left(9C^{1234} + 5S^{1234} - \delta_2^{12}\delta_3^{34} - 3\Upsilon^{1234}\right) s_2^1 s_2^2 s_3^3 s_3^4.\end{aligned} \tag{B.9}$$

This also happened in all other cases treated so far, where four operators of equal weight ($k = 2, 3, 4$) were considered [8 − 10].[6]

C. *D*-functions

In the evaluation of the graphs contributing to the four-point functions, an important rôle is played by the so-called *D-functions*. These D-functions correspond to a quartic interaction of scalar fields [12]. The D-functions related to AdS_5 are defined by

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_1, x_2, x_3, x_4) = \int \frac{d^5 z}{z_0^5} K_{\Delta_1}(z, x_1) K_{\Delta_2}(z, x_2) K_{\Delta_3}(z, x_3) K_{\Delta_4}(z, x_4), \tag{C.1}$$

where K_Δ is the bulk-to-boundary propagator for scalar fields, defined by

$$K_\Delta(z, \vec{x}) = \left(\frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2)}\right)^\Delta.$$

It is possible to express the D-functions in terms of the conformal cross-ratios s and t. Introducing the notation $\overline{D}$, we have based on the conformal symmetries

$$\frac{\prod_{i=1}^4 \Gamma(\Delta_i)}{\Gamma(\Sigma - 2)} \frac{2}{\pi^2} D_{\Delta_1\Delta_2\Delta_3\Delta_4} = \frac{(x_{14}^2)^{\Sigma-\Delta_1-\Delta_4}(x_{34}^2)^{\Sigma-\Delta_3-\Delta_4}}{(x_{13}^2)^{\Sigma-\Delta_4}(x_{24}^2)^{\Delta_2}} \overline{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(s, t), \tag{C.2}$$

where $\Sigma = \frac{1}{2}\sum_{i=1}^4 \Delta_i$. For $\Delta_i = 1$, this expression becomes

$$\overline{D}_{1111}(s, t) = \Phi(s, t), \tag{C.3}$$

where $\Phi(s, t)$ is the one-loop (box) integral as a function of the conformal cross-ratios [23]. There are a number of relations for these $\overline{D}$-functions, which can be used to simplify expressions and check crossing symmetry. We do not need them here, but see ref. [9] for a list.

[6] After the completion of this calculation, we learned from [14], that the same result was proved that these four-derivative couplings have to vanish for the AdS/CFT correspondence to be consistent. This is a so-called sub-subextremal case, where $k_1 = k_2 + k_3 + k_4 - 4$. If this coupling were non-zero, the associated contact diagram would lead to divergences. The calculation done there is in the same spirit as this one.

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By considering relations for derivatives of the originial D-functions, it is possible to derive the following relations for the $\overline{D}$-functions [9]

$$\begin{aligned}
\overline{D}_{\Delta_1+1\Delta_2+1\Delta_3\Delta_4} &= -\partial_s \overline{D}_{\Delta_1\Delta_2\Delta_3\Delta_4} \\
\overline{D}_{\Delta_1\Delta_2\Delta_3+1\Delta_4+1} &= (\Delta_3+\Delta_4-\Sigma-s\partial_s)\overline{D}_{\Delta_1\Delta_2\Delta_3\Delta_4} \\
\overline{D}_{\Delta_1\Delta_2+1\Delta_3+1\Delta_4} &= -\partial_t \overline{D}_{\Delta_1\Delta_2\Delta_3\Delta_4} \\
\overline{D}_{\Delta_1+1\Delta_2\Delta_3\Delta_4+1} &= (\Delta_1+\Delta_4-\Sigma-t\partial_t)\overline{D}_{\Delta_1\Delta_2\Delta_3\Delta_4} \\
\overline{D}_{\Delta_1\Delta_2+1\Delta_3\Delta_4+1} &= (\Delta_2+s\partial_s+t\partial_t)\overline{D}_{\Delta_1\Delta_2\Delta_3\Delta_4} \\
\overline{D}_{\Delta_1+1\Delta_2\Delta_3+1\Delta_4} &= (\Sigma-\Delta_4+s\partial_s+t\partial_t)\overline{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}.
\end{aligned} \tag{C.4}$$

Starting with (C.3) and subsequently applying these relations, it is possible to assign to each D-function a differential operator $\overline{\mathbf{D}}$, such that

$$\overline{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(s,t) = \overline{\mathbf{D}}_{\Delta_1\Delta_2\Delta_3\Delta_4}\Phi(s,t), \tag{C.5}$$

as long as each Δ_1 is an integer, and their sum is even. Note that there is some arbitrariness in defining $\overline{\mathbf{D}}$, because there are different combinations of the relations in eqs. (C.4) that one can use to find a particular differential operator.

The action of the partial derivatives on $\Phi(s,t)$ is known [6], they are given by

$$\begin{aligned}
\partial_s\Phi(s,t) &= \frac{1}{\lambda^2}\left(\Phi(s,t)(1-s+t)+2\ln s-\frac{s+t-1}{s}\ln t\right) \\
\partial_t\Phi(s,t) &= \frac{1}{\lambda^2}\left(\Phi(s,t)(1-t+s)+2\ln t-\frac{s+t-1}{t}\ln s\right),
\end{aligned} \tag{C.6}$$

where $\lambda = \sqrt{(1-s-t)^2-4st}$. Using this together with eq. (C.5), it is possible to express each combination of D-functions into an expression involving only $\Phi(s,t)$. This is the method we will use to check the partial non-renormalization of the four-point function.

D. Exchange graphs

In this section we generalize the method of [24] and appendix E of [9] to calculate the z-integrals in exchange diagrams. Our results include couplings to scalar fields of different mass. Together with the previous results cited above, all exchange diagrams contributing to *arbitrary* four-point functions of $\frac{1}{2}$-BPS operators can be calculated.

We briefly review the method used: First, conformal symmetry is used to bring the z-integral into a simpler form. The basic idea is then to use the wave equation for the propagator in the z-integral. Based on conformal invariance an ansatz for the z-integral is proposed. Then, the wave equation is applied to the integrand of the z-integral and on the ansatz. This leads to a system of differential equations, from which we can solve for the z-integral.

D.1 Vector exchange

Here we generalize the calculation of the vector z-integral done in [24] to include the case when the vector field couples to scalar fields of different mass. The z-integral is given by

$$A_\mu(\omega, \vec{x}_1, \vec{x}_3) = \int \frac{d^{d+1}z}{z_0^{d+1}} G_{\mu\nu'}(\omega, z) g^{\nu'\rho'}(z) K_{\Delta_1}(z, \vec{x}_1) \frac{\overleftrightarrow{\partial}}{\partial z^{\rho'}} K_{\Delta_3}(z, \vec{x}_3),$$

where $K_\Delta(z, \vec{x})$ is the canonically normalized bulk-to-boundary propagator for a scalar field s_Δ.

The gauge boson propagator $G_{\mu\nu'}(\omega, z)$ with mass M in AdS_5 satisfies the defining wave equation:

$$-\nabla^\mu \nabla_{[\mu} G_{\rho]\nu'} + M^2 G_{\rho\nu'} = g_{\rho\nu'} \delta(\omega, z) + \partial_\rho \partial_{\nu'} \Lambda(u), \tag{D.1}$$

where [..] denotes anti-symmetrization. We can drop the gauge term $\Lambda(u)$, since this is only necessary for couplings to the massless gauge boson. However, the gauge boson couples only to conserved currents and we refer to [24] for this case.

Using $A_{\mu\nu}(\omega, \vec{x}_1, \vec{x}_3) = A_{\mu\nu}(\omega - \vec{x}_1, 0, \vec{x}_{13})$, where $\vec{x}_{13} \equiv \vec{x}_3 - \vec{x}_1$, and performing the conformal inversion $\omega'_\mu = \omega_\mu/\omega^2$, $x'_\mu = x_\mu/x^2$ the vector z integral takes the following form

$$A_\mu(\omega, \vec{x}_1, \vec{x}_3) = |\vec{x}_{13}|^{-2\Delta_3} \frac{1}{\omega^2} J_{\mu\nu}(\omega) I_\mu(\omega' - \vec{x}'_{13}), \tag{D.2}$$

where $J_{\mu\nu}(\omega) = \delta_{\mu\nu} - \frac{2}{\omega^2}\omega_\mu\omega_\nu$, and

$$I_\mu(\omega) = \int \frac{d^{d+1}z}{z_0^{d+1}} G_\mu{}^{\nu'}(\omega, z) z_0^{\Delta_1} \frac{\overleftrightarrow{\partial}}{\partial z^{\nu'}} \left(\frac{z_0}{z^2}\right)^{\Delta_3}. \tag{D.3}$$

We write the following ansatz, with $\Delta_{13} = \Delta_1 - \Delta_3$,

$$I_\mu(\omega) = \omega_0^{\Delta_{13}} \frac{\omega_\mu}{\omega^2} f(t) + \omega_0^{\Delta_{13}} \frac{\delta_{\mu 0}}{\omega_0} h(t), \tag{D.4}$$

where $t = \omega_0^2/\omega^2$.

To find the unknown scalar functions $f(t)$ and $h(t)$ we equate eqs. (D.3) and (D.4) and apply the differential operator in eq. (D.1) to both sides. For eq. (D.3) we obtain

$$-\nabla^\mu \nabla_{[\mu} I_{\rho]} + M^2 I_\rho = -\omega_0^{\Delta_{13}} \frac{\omega_\rho}{\omega^2} 2\Delta_3 t^{\Delta_3} - \omega_0^{\Delta_{13}} \frac{\delta_{\rho 0}}{\omega_0} \Delta_{13} t^{\Delta_3}. \tag{D.5}$$

For eq. (D.4) the Maxwell term becomes

$$-\nabla^\mu \nabla_{[\mu} I_{\rho]} = \omega^2 \delta_{\mu\lambda} \partial_\lambda T_{\rho\mu} + \omega^2 \delta_{\mu\lambda} \Gamma^\kappa_{\rho\lambda} T_{\kappa\mu} + \omega^2 \delta_{\mu\lambda} \Gamma^\kappa_{\lambda\mu} T_{\rho\kappa}, \tag{D.6}$$

where we defined

$$T_{\rho\mu} := \partial_{[\mu} I_{\rho]} = \left(\omega_\rho \frac{\delta_{\mu 0}}{\omega_0} - \omega_\mu \frac{\delta_{\rho 0}}{\omega_0}\right) \left(\frac{2\omega_0^{\Delta_{13}+2}}{(\omega^2)^2} \left(f' + h'\right) + \Delta_{13} \frac{\omega_0^{\Delta_{13}}}{\omega^2} f(t)\right). \tag{D.7}$$

We omit the tedious but straightforward calculation of these terms.

Now, substituting the result in the l.h.s. of equation (D.5), we find the equation in terms of f and h. To solve it, we equate the corresponding contributions of the tensor structures $\frac{\delta_{\rho 0}}{\omega_0}$ and $\frac{\omega_\rho}{\omega^2}$ to the l.h.s. and r.h.s., and obtain the following coupled system of inhomogeneous differential equations:

$$2\Delta_{13}t^2(f'+h') + \Delta_{13}^2 tf + M^2 h = -\Delta_{13}t^{\Delta_3}, \tag{D.8}$$

$$\begin{aligned} 4t^2(t-1)\left(f''+h''\right) + 2t[t(4+\Delta_{13}) + (d-4-2\Delta_{13})]f' \\ +2t[4t+d-4-\Delta_{13}]h' + [\Delta_{13}(2t+d-2-\Delta_{13}) + M^2]f = -2\Delta_3 t^{\Delta_3}. \end{aligned} \tag{D.9}$$

To solve the differential equations we assume power series expansions

$$f(t) = \sum_k a_k t^k, \qquad h(t) = \sum_k b_k t^k, \tag{D.10}$$

with $k_{\min} \leq k \leq k_{\max}$.

Substituting eq. (D.10) in eqs. (D.8) and (D.9); and intersecting the equations we find b_k and b_{k+1} in terms of a_k and a_{k+1}:

$$\begin{aligned} b_k &= -\frac{2k+\Delta_{13}}{2k}a_k + \frac{M^2[2(-2k-4+d)(k+1) + \Delta_{13}(4k-6+d-\Delta_{13}) + M^2]}{4(k+1)k[\Delta_{13}(-2k-4+d-\Delta_{13}) - M^2]}a_{k+1} \\ &\quad + \frac{\Delta_{13}(2\Delta_3+2-d+\Delta_{13}) - M^2}{2(\Delta_{13}-1)[(-2\Delta_3-2+d-\Delta_{13}) - M^2]}\delta_{k,\Delta_3-1}, \\ b_{k+1} &= -\frac{\Delta_{13}[2(k+1)(-2k-4+d) + \Delta_{13}(-4k-6+d-\Delta_{13}) + M^2]}{2(k+1)[\Delta_{13}(-2k-4+d-\Delta_{13}) - M^2]}a_{k+1}. \end{aligned} \tag{D.11}$$

The solution is found if we substitute the second equation in the first one:

$$\begin{aligned} a_k &= 0 \qquad \text{for} \qquad k \geq \Delta_3 \\ a_{\Delta_3-1} &= \frac{\Delta_{13}(d-2\Delta_3) - \Delta_{13}^2 - M^2}{2M^2(\Delta_3-1+\Delta_{13})}, \\ a_k &= \frac{[\Delta_{13}^2 + \Delta_{13}(2k+2-d)][2(k+1)(2k+4-d) + \Delta_{13}(4k+6-d) + \Delta_{13}^2 - M^2]}{4(k+1)(k+\Delta_{13})[\Delta_{13}(2k+4-d) + \Delta^2 + M^2]}a_{k+1} \end{aligned} \tag{D.12}$$

From this we see that the series terminates at

$$0 \leq k_{\min} = \frac{d-2-2\Delta_{13}}{4} + \frac{1}{4}\sqrt{(d-2)^2 + 4M^2} \leq k_{\max} = \Delta_3 - 1,$$

provided that $k_{\max} - k_{\min}$ is an integer and ≥ 0.

One can show using table III in [25], where $M^2 = l^2 - 1$ with $l \in \mathbb{Z}^+$, that the terminating condition is always satisfied by Type IIB supergravity compactified in $AdS_5 \times S_5$ due to the SO(6) selection rules [26].[7]

[7] It is worth to recall that the marginal case, when the equality holds and the series doesn't terminate, is also allowed by the SO(6) selection rules. We consider here only the terminating case. Note also that the case $M^2 = 0$ requires $\Delta_1 + \Delta_3 - d \in \mathbb{N}$ for terminating of the series.

Finally, to recover the vector z-integral in terms of the original coordinates, we must transform the coordinates back. This amounts to

$$\begin{aligned} \omega_0' &\to \frac{\omega_0}{\omega_0^2 + (\vec{\omega} - \vec{x}_1)^2}, \\ t = \frac{{\omega_0'}^2}{(\omega' - \vec{x}_{31}')^2} &\to q = \vec{x}_{31}^2 \frac{\omega_0}{\omega_0^2 + (\vec{\omega} - \vec{x}_1)^2} \frac{\omega_0}{\omega_0^2 + (\vec{\omega} - \vec{x}_3)^2}, \\ \frac{1}{\omega^2} J_{\mu\lambda}(\omega) \frac{(\omega' - \vec{x}_{31}')_\lambda}{(\omega' - \vec{x}_{31}')^2} &\to Q_\mu := \frac{(\omega - \vec{x}_3)_\mu}{(\omega - \vec{x}_3)^2} - \frac{(\omega - \vec{x}_1)_\mu}{(\omega - \vec{x}_1)^2} \\ \frac{1}{\omega^2} J_{\mu\lambda}(\omega) \frac{\delta_{\lambda 0}}{\omega_0'} &\to R_\mu := \frac{\delta_{\mu 0}}{\omega_0} - 2 \frac{(\omega - \vec{x}_1)_\mu}{(\omega - \vec{x}_1)^2}. \end{aligned} \tag{D.13}$$

In the case we need in this paper, $\Delta_1 = 3$, $\Delta_3 = 2$ and $M^2 = 3$, we find

$$A_\mu(\omega, \vec{x}_1, \vec{x}_3) = \frac{1}{\vec{x}_{31}^2} \left[-\frac{1}{12} K_1(\omega, \vec{x}_3) \nabla_\mu K_2(\omega, \vec{x}_1) + \frac{1}{6} K_2(\omega, \vec{x}_1) \nabla_\mu K_1(\omega, \vec{x}_3) \right]. \tag{D.14}$$

The ω-integral can then be calculated straightforwardly.

D.2 Symmetric tensor exchange

Here we extend the computation in [9] using the method of [24] for the massive symmetric tensor z-integral when the tensor field is coupled to scalar fields of different mass. In this case the stress energy tensor $T_{\mu\nu}$ is not covariantly conserved and has the form

$$T_{\mu\nu} = \frac{1}{2} \nabla_{(\mu} s_{\Delta_1} \nabla_{\nu)} s_{\Delta_3} - \frac{1}{2} g_{\mu\nu} \left(\nabla^\rho s_{\Delta_1} \nabla_\rho s_{\Delta_3} + \frac{1}{2} (m_{\Delta_1}^2 + m_{\Delta_3}^2 - f) s_{\Delta_1} s_{\Delta_3} \right), \tag{D.15}$$

where s_Δ denotes a scalar field of mass squared $m_\Delta^2 = \Delta(\Delta - 4)$, and f is the mass squared of the tensor.

The z-integral describing the exchange of a massive symmetric tensor is given by

$$A_{\mu\nu}(\omega, \vec{x}_1, \vec{x}_3) := \int \frac{d^{d+1}z}{z_0^{d+1}} G_{\mu\nu\mu'\nu'}(\omega, z) T^{\mu'\nu'}(z, \vec{x}_1, \vec{x}_3). \tag{D.16}$$

The tensor $T^{\mu\nu}(z, \vec{x}_1, \vec{x}_3)$ has the form

$$\begin{aligned} T^{\mu\nu}(z, \vec{x}_1, \vec{x}_3) = {} & \frac{1}{2} \nabla^{(\mu} K_{\Delta_1}(z, \vec{x}_1) \nabla^{\nu)} K_{\Delta_3}(z, \vec{x}_3) - \frac{1}{2} g^{\mu\nu} \left[\nabla_\rho K_{\Delta_1}(z, \vec{x}_1) \nabla^\rho K_{\Delta_3}(z, \vec{x}_3) \right. \\ & \left. + \frac{1}{2} \left(m_{\Delta_1}^2 + m_{\Delta_3}^2 - f \right) K_{\Delta_1}(z, \vec{x}_1) K_{\Delta_3}(z, \vec{x}_3) \right], \end{aligned} \tag{D.17}$$

where (..) denotes symmetrization.

The Ricci form of the wave equation for the bulk-to-bulk propagator $G_{\mu\nu\mu'\nu'}(\omega, z)$ for the massive symmetric tensor field is

$$W_{\mu\nu}{}^{\lambda\rho}[G_{\lambda\rho\mu'\nu'}] = \left(g_{\mu\mu'} g_{\nu\nu'} + g_{\mu\nu'} g_{\nu\mu'} - \frac{2}{d-1} g_{\mu\nu} g_{\mu'\nu'} \right) \delta(\omega, z). \tag{D.18}$$

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Pure gauge terms are omitted, because they are not needed in the case of massive tensors. The graviton only couples to conserved currents, and we refer to appendix E of [9] for this case.

To solve the tensor z-integral we use again $A_{\mu\nu}(\omega, \vec{x}_1, \vec{x}_3) = A_{\mu\nu}(\omega - \vec{x}_1, 0, \vec{x}_{13})$ and perform the conformal inversion $\omega'_\mu = \omega_\mu/\omega^2$, $\vec{x}'_\mu = \vec{x}_\mu/x^2$ on eq. (D.16) to obtain

$$A_{\mu\nu}(\omega, \vec{x}_1, \vec{x}_3) = |\vec{x}_{13}|^{-2\Delta_3} \frac{1}{(\omega^2)^2} J_{\mu\lambda}(\omega) J_{\nu\rho}(\omega) I_{\lambda\rho}(\omega' - \vec{x}'_{13}), \tag{D.19}$$

where

$$\begin{aligned} I_{\mu\nu}(\omega) = \int \frac{d^{d+1}z}{z_0^{d+1}} G_{\mu\nu}{}^{\mu'\nu'}(\omega, z) \Bigg\{ &\nabla_{(\mu'} z_0^{\Delta_1} \nabla_{\nu')} \left(\frac{z_0}{z^2}\right)^{\Delta_3} \\ &- g_{\mu'\nu'} \left[\nabla_{\rho'} z_0^{\Delta_1} \nabla^{\rho'} \left(\frac{z_0}{z^2}\right)^{\Delta_3} + \frac{1}{2}(m_1^2 + m_2^2 - f) z_0^{\Delta_1} \left(\frac{z_0}{z^2}\right)^{\Delta_3} \right] \Bigg\}. \end{aligned} \tag{D.20}$$

We write the following ansatz

$$\begin{aligned} I_{\mu\nu}(\omega) &= \omega_0^{\Delta_{13}} g_{\mu\nu} h(t) + \omega_0^{\Delta_{13}} P_\mu P_\nu \phi(t) + \omega_0^{\Delta_{13}} \nabla_\mu \nabla_\nu X(t) + \omega_0^{\Delta_{13}} \nabla_{(\mu} \left(P_{\nu)} Y(t)\right) \\ &= \omega_0^{\Delta_{13}} \tilde{I}_{\mu\nu}(\omega), \end{aligned} \tag{D.21}$$

where $P_\mu := \delta_{\mu 0}/\omega_0$; $h(t)$, $\phi(t)$, $X(t)$, $Y(t)$ are four unknown scalar functions, and $\tilde{I}_{\mu\nu}(\omega)$ is the ansatz used in [9] for the case $\Delta_1 = \Delta_3$. To find the functions we first have to equate eqs. (D.20) and (D.21) and apply the modified Ricci operator on both sides. For eq. (D.21) we first note that

$$\begin{aligned} \nabla_\rho \nabla_\sigma I_{\mu\nu} &= \nabla_\rho \nabla_\sigma \left(\omega_0^{\Delta_{13}} \tilde{I}_{\mu\nu}\right) \\ &= \omega_0^{\Delta_{13}} \left(\nabla_\rho \nabla_\sigma \tilde{I}_{\mu\nu}\right) + \left(\nabla_{(\rho} \omega_0^{\Delta_{13}}\right) \left(\nabla_{\sigma)} \tilde{I}_{\mu\nu}\right) + \left(\nabla_\rho \nabla_\sigma \omega_0^{\Delta_{13}}\right) \tilde{I}_{\mu\nu}. \end{aligned}$$

This identity allows us to write

$$W_{\mu\nu}{}^{\lambda\rho}[I_{\lambda\rho}] = \omega_0^{\Delta_{13}} \left(W_{\mu\nu}{}^{\lambda\rho}[\tilde{I}_{\rho\sigma}]\right) + H_{\mu\nu} + N_{\mu\nu}, \tag{D.22}$$

where

$$\begin{aligned} H_{\mu\nu} &= \left[-\left(\nabla^2 \omega_0^{\Delta_{13}}\right) \tilde{I}_{\rho\sigma} - \left(\nabla_\mu \nabla_\nu \omega_0^{\Delta_{13}}\right) I^\sigma{}_\sigma + \left(\nabla_\mu \nabla^\sigma \omega_0^{\Delta_{13}}\right) I_{\sigma\nu} + \left(\nabla_\nu \nabla^\sigma \omega_0^{\Delta_{13}}\right) I_{\mu\sigma} \right], \\ N_{\mu\nu} &= \Big[-2 \left(\nabla_\sigma \omega_0^{\Delta_{13}}\right) \left(\nabla^\sigma \tilde{I}_{\rho\sigma}\right) - \left(\nabla_{(\mu} \omega_0^{\Delta_{13}}\right) \left(\nabla_{\nu)} I^\sigma{}_\sigma\right) \\ &\quad + g^{\sigma\kappa} \left(\nabla_{(\mu} \omega_0^{\Delta_{13}}\right) \left(\nabla_{\kappa)} I_{\sigma\nu}\right) + g^{\sigma\kappa} \left(\nabla_{(\nu} \omega_0^{\Delta_{13}}\right) \left(\nabla_{\kappa)} I_{\mu\sigma}\right) \Big]. \end{aligned}$$

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The first term in eq. (D.22) was calculated in appendix E of [9]. For the $H_{\mu\nu}$ and $N_{\mu\nu}$ we obtain the following formulae

$$
\begin{aligned}
H_{\mu\nu} = {\omega_0}^{\Delta_{13}}\Delta_{13}\Big\{ & g_{\mu\nu}\big[(2d-1-\Delta_{13})h+\phi+4t^2(1-t)X''+2t(2-2t-d)X' \\
& \qquad +4t(1-t)Y'+[4(-d+1)+2\Delta_{13}]Y\big] \\
& +P_\mu P_\nu\big[(\Delta_{13}+1)(-d+1)h+(d-1)\phi+(\Delta_{13}+1)4t^2(1-t)X'' \\
& \qquad +(\Delta_{13}+1)2t(2-4t+d)X'+4t(d-1)Y'+(4+2\Delta_{13})(d-1)Y\big] \\
& +(d-2-\Delta_{13})\nabla_\mu\nabla_\nu X \\
& +P_{(\mu}\frac{\omega_{\nu)}}{\omega^2}\big[(\Delta_{13}+1)4t^2(1-t)X''+(\Delta_{13}+1)2t(4t-3)X'+2t(-d+1)Y'\big]\Big\}, \\
N_{\mu\nu} = {\omega_0}^{\Delta_{13}}\Delta_{13}\Big\{ & g_{\mu\nu}\big[4t(t-1)h'-2\phi+4t(t-1)X'+4t(1-t)Y'-4Y\big] \\
& +P_\mu P_\nu\big[4t(-d+1)h'+2(-d+1)\phi+4t(-d+1)X' \\
& \qquad +8t^2(t-1)Y''+4t(4t-3)Y'+4(-d+1)Y\big] \\
& +2\nabla_\mu\nabla_\nu Y \\
& +P_{(\mu}\frac{\omega_{\nu)}}{\omega^2}\big[2t(d-1)h'+2t(d-1)X'+8t^2(1-t)Y''+2t(-d+8-8t)Y'\big]\Big\}.
\end{aligned} \tag{D.23}
$$

For eq. (D.20), due to the defining wave equation for the propagator of the symmetric tensor field eq. (D.18), we obtain

$$
\begin{aligned}
{W_{\mu\nu}}^{\lambda\rho}[I_{\lambda\rho}] = & \omega_0^{\Delta_{13}} g_{\mu\nu}\frac{2}{d-1}(m_1^2+m_2^2-f)t^{\Delta_3}+\omega_0^{\Delta_{13}}P_\mu P_\nu 4\Delta_1\Delta_3 t^{\Delta_3} \\
& -\omega_0^{\Delta_{13}}P_{(\mu}\frac{\omega_{\nu)}}{\omega^2}4\Delta_1\Delta_3 t^{\Delta_3}.
\end{aligned} \tag{D.24}
$$

From eqs. (D.22) and (D.23) the basic eq. (D.24) can be written in terms of the four unknown functions in eq. (D.21). To determine them we first equate the terms involving $\nabla_\mu\nabla_\nu$ in both sides:

$$\nabla_\mu\nabla_\nu[-3h-\phi+(d\Delta_{13}-2\Delta_{13}-\Delta_{13}^2+f)X+2\Delta_{13}Y]=0. \tag{D.25}$$

Now we equate the coefficients of the tensor $P_{(\mu}\frac{\omega_{\nu)}}{\omega^2}$ to get

$$
\begin{aligned}
& 2\Delta_{13}(d-1)h'+4t(t-1)\phi''+8t\phi' \\
& +(\Delta_{13}^2+\Delta_{13})4t(t-1)X''+[\Delta_{13}^2 2(4t-3)+\Delta_{13}2(4t-4+d)]X' \\
& +\Delta_{13}8t(1-t)Y''+2[-f+\Delta_{13}(8-8t-2d)]Y'=-4\Delta_1\Delta_3 t^{\Delta_3-1}.
\end{aligned} \tag{D.26}
$$

This equation can be integrated to give

$$
\begin{aligned}
& 4t(t-1)\phi'+4\phi+(\Delta_{13}^2+\Delta_{13})4t(t-1)X'+2(\Delta_{13}d-2\Delta_{13}-\Delta_{13}^2)X \\
& +\Delta_{13}8t(1-t)Y'+2(-f+4\Delta_{13}-2d\Delta_{13})Y+2\Delta_{13}(d-1)h=-4\Delta_1 t^{\Delta_3}+c_1,
\end{aligned} \tag{D.27}
$$

where c_1 is an integration constant. Equating the coefficients of $P_\mu P_\nu$, we get

$$
\begin{aligned}
&4t\Delta_{13}(-d+1)h' + (\Delta_{13}^2+\Delta_{13})(-d+1)h + 4t^2(1-t)\phi'' - 8t^2\phi' + [\Delta_{13}(-d+1)+f]\phi \\
&\quad + (\Delta_{13}^2 + \Delta_{13})4t^2(1-t)X'' + [\Delta_{13}^2 2t(2-4t+d) + \Delta_{13}2t(4-4t-d)]X' \\
&\qquad + \Delta_{13}8t^2(t-1)Y'' + 4t[f + \Delta_{13}(4t-4+d)]Y' + 2[f + \Delta_{13}^2(d-1)]Y = 4\Delta_1\Delta_3 t^{\Delta_3}.
\end{aligned}
$$

Substituting here eq. (D.26) we find

$$
\begin{aligned}
&2t\Delta_{13}(-d+1)h' + (\Delta_{13}^2 + \Delta_{13})(-d+1)h + [f + \Delta_{13}(-d+1)]\phi \\
&\qquad + \Delta_{13}^2 2t(d-1)X' + 2tfY' + 2[f + \Delta_{13}^2(d-1)]Y = 0.
\end{aligned}
\tag{D.28}
$$

Finally, we equate the coefficients of $g_{\mu\nu}$

$$
\begin{aligned}
&4t^2(t-1)h'' + [4t(t+1) + \Delta_{13}4t(t-1)]h' + \left[\frac{8}{3}(f+3) + 2d\Delta_{13} - \Delta_{13} - \Delta_{13}^2\right] h \\
&\quad + 4t(t-1)\phi' + \left[-\Delta_{13} + \frac{1}{3}(f+24)\right]\phi \\
&\quad + \left[\frac{f}{3} + \Delta_{13}\right] 4t^2(1-t)X'' + \left[-\frac{4f}{3}t(t+1) - 2td\Delta_{13}\right] X' \\
&\quad + \left[\frac{f}{3} + 2\Delta_{13}\right] 4t(1-t)Y' + \left[-\frac{14f}{3} + 2\Delta_{13}^2 - 4\Delta_{13}d\right] Y = \frac{2}{d-1}(m_1^2 + m_2^2 - f)t^{\Delta_3}.
\end{aligned}
\tag{D.29}
$$

Eqs. (D.25), (D.27), (D.28) and (D.29) form a system of four differential equations whose solution, regular as $t \to 0$ and $t \to 1$, determines the tensor z-integral. We solve them for our specific case: $d = 4$, $\Delta_1 = 3$, $\Delta_3 = 2$ and $f = 5$; which implies $\Delta_{13} = 1$, $m_1^2 = -3$ and $m_3^2 = -4$:

$$
\nabla_\mu\nabla_\nu(-3h - \phi + 6X + 2Y) = 0, \tag{D.30}
$$

$$
6h + 4t(t-1)\phi' + 4\phi + 8t(t-1)X' + 2X - 8t(t-1)Y' - 18Y = -12t^2 + c_1, \tag{D.31}
$$

$$
6th' + 6h - 2\phi - 6tX' - 10tY' - 16Y = 0, \tag{D.32}
$$

$$
\begin{aligned}
&12t^2(t-1)h'' + 24t^2h' + 82h + 12t(t-1)\phi' + 26\phi \\
&\quad + 32t^2(1-t)X'' + [-20t^2 - 44t]X' + 44t(t-1)X - 112Y = -24t^2.
\end{aligned}
\tag{D.33}
$$

From eq. (D.30) we pick up the trivial solution

$$
h = -\frac{1}{3}\phi + 2X + \frac{2}{3}Y \tag{D.34}
$$

and substitute it in eq. (D.32) to obtain

$$
2t(\phi' - 3X' + 3Y') + 4(\phi - 3X + 3Y) = 0.
$$

This equation can be trivially integrated to give

$$
\phi - 3X + 3Y = c_2 t^{-2}. \tag{D.35}
$$

This is regular as $t \to 0$ for $c_2 = 0$. Substituting h and ϕ in terms of X and Y in eq. (D.31), we get

$$20t(t-1)(X' - Y') + 20(X - Y) = -12t^2 + c_1.$$

The solution consistent with the asymptotic behavior is given by

$$X = Y - \frac{6}{10}t + \frac{c_1}{20}. \tag{D.36}$$

Finally, we substitute eqs. (D.34), (D.35) and (D.36) in eq. (D.33) and find

$$Y(t) = \frac{18t - 3c_1}{40}.$$

Substituting back, we find the remaining three functions

$$X(t) = \frac{-6t - c_1}{40}, \qquad \phi(t) = \frac{-36t + 3c_1}{20}, \qquad h(t) = \frac{12t - 3c_1}{20}.$$

Upon substitution of the four functions in eq. (D.21), all the terms proportional to the integration constant c_1 cancel and the tensor integral becomes

$$I_{\mu\nu}(\omega) = \omega_0 \left(\frac{3}{5} t g_{\mu\nu} - \frac{9}{5} t P_\mu P_\nu - \frac{3}{20} \nabla_\mu \nabla_\nu t + \frac{9}{20} \nabla_{(\mu} \left(P_{\nu)} t \right) \right).$$

Working out the derivatives, we find

$$I_{\mu\nu}(\omega) = -\frac{6}{5}\omega_0 t \frac{\omega_\mu \omega_\nu}{(\omega^2)^2}.$$

In terms of the original coordinates, the z-integral describing the exchange by a symmetric tensor field of mass squared $f = 5$ is

$$A_{\mu\nu}(\omega, \vec{x}_1, \vec{x}_3) = -\frac{1}{\vec{x}_{31}^2} \frac{6}{5} Q_\mu Q_\nu K_2(\omega, \vec{x}_1) K_1(\omega, \vec{x}_3),$$

where Q_μ is defined in eq. (D.13).

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